FOCUS Curvature Objective Function

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General

FOCUS can optimize for a coil's curvature through the curvature objective function given in (1). This objective function can constrain the coil's maximum curvature by using the penalty functions p_1 and p_2 and it can optimize for an average curvature quantity. A coil is parameterized as $\boldsymbol{r}(\zeta)$ with $\zeta \in [0, 2\pi]$.

$$f_\kappa = \frac{1}{N_c} \sum_{i=1}^{N_c} \frac{1}{L_i} \int_0^{2\pi} \left(p_{1,2}(\kappa_i, \kappa_0) + \sigma \kappa_i^\gamma \right) \left| \boldsymbol{r_i}' \right| d\zeta \tag{1}$$

$$\kappa_0 \geq 0 \qquad \sigma \geq 0 \qquad \gamma \geq 1$$

$$p_1(\kappa, \kappa_0) = H_{\kappa_0}(\kappa) \left(\cosh(\alpha(\kappa - \kappa_0)) - 1 \right)^2 \tag{2}$$

$$\alpha \geq 0 \qquad \kappa_0 \geq 0$$

$$p_2(\kappa, \kappa_0) = H_{\kappa_0}(\kappa) \left(\alpha(\kappa - \kappa_0) \right)^\beta \tag{3}$$

$$\alpha \geq 0 \qquad \kappa_0 \geq 0 \qquad \beta \geq 2$$

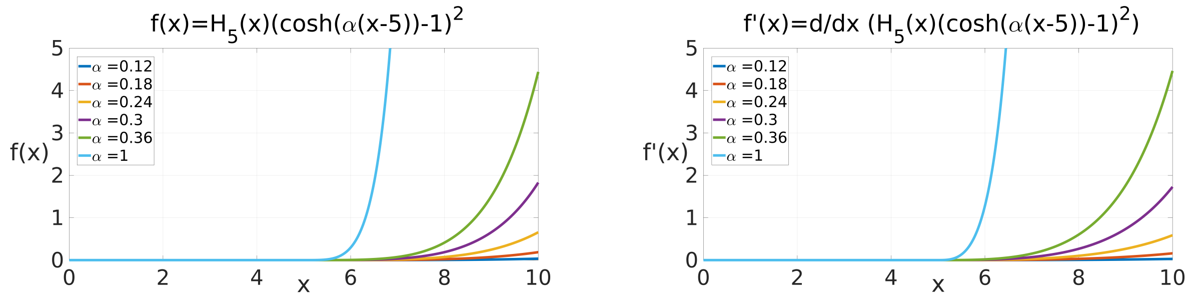
$$H_{\kappa_0}(\kappa_i) \equiv H(\kappa_i - \kappa_0) = \begin{cases} 0, & \kappa_i < \kappa_0 \\ \frac{1}{2}, & \kappa_i = \kappa_0 \\ 1, & \kappa_i > \kappa_0 \end{cases} \tag{4}$$

$$\kappa = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} \tag{5}$$

First Derivatives

First derivatives of the curvature objective function, (1) are now given. Let λ_j be an optimization variable that defines the position of the j-th coil, $\boldsymbol{r_j}$ where $j \in \{1, 2, ..., N_c\}$. The coil length functional, L and its derivatives can be found in the length objective function documentation.

$$\begin{aligned} \frac{\partial f_\kappa}{\partial \lambda_j} = \frac{-1}{N_c L_j^2} \frac{\partial L_j}{\partial \lambda_j} \int_0^{2\pi} \left(p_{1,2} + \sigma \kappa_j^\gamma \right) \left| \boldsymbol{r_j}' \right| d\zeta + \frac{1}{N_c L_j} \int_0^{2\pi} \left(\frac{\partial p_{1,2}}{\partial \kappa_j} + \sigma \gamma \kappa_j^{\gamma - 1} \right) \frac{\partial \kappa_j}{\partial \lambda_j} \left| \boldsymbol{r_j}' \right| d\zeta + \\ \frac{1}{N_c L_j} \int_0^{2\pi} \left(p_{1,2} + \sigma \kappa_j^\gamma \right) \left| \boldsymbol{r_j}' \right|^{-1} \boldsymbol{r_j}' \cdot \frac{\partial \boldsymbol{r_j}'}{\partial \lambda_j} d\zeta \end{aligned} \tag{6}$$



(a) (b)

Figure 1: The function p_1 is plotted in (a) where the variable κ is changed to x and the value of κ_0 is set to 5. Multiple values of the penalty function variable, α are plotted. The derivative is plotted in (b).

$$\frac{\partial p_1}{\partial \kappa} = H_{\kappa_0}(\kappa)\, 2\alpha \left(\cosh(\alpha(\kappa - \kappa_0)) - 1 \right) \sinh(\alpha(\kappa - \kappa_0)) \tag{7}$$

$$\frac{\partial p_2}{\partial \kappa} = H_{\kappa_0}(\kappa) \beta\alpha \left(\alpha(\kappa - \kappa_0)\right)^{\beta-1} \tag{8}$$

$$\frac{\partial \kappa}{\partial \lambda} = -3 \left|\boldsymbol{r}'\right|^{-5} \boldsymbol{r}' \cdot \frac{\partial \boldsymbol{r}'}{\partial \lambda} \left|\boldsymbol{r}' \times \boldsymbol{r}''\right| + \left|\boldsymbol{r}'\right|^{-3} \left|\boldsymbol{r}' \times \boldsymbol{r}''\right|^{-1} \left(\boldsymbol{r}' \times \boldsymbol{r}''\right) \cdot \left(\frac{\partial \boldsymbol{r}'}{\partial \lambda} \times \boldsymbol{r}'' + \boldsymbol{r}' \times \frac{\partial \boldsymbol{r}''}{\partial \lambda} \right) \tag{9}$$

Notes

The above derivative equations do not use functional derivatives. Functional derivatives for this curvature objective function are very complicated and due to that complexity are not implemented in FOCUS. FOCUS has the above derivatives implemented for a Fourier parameterized coil. There is some development work required if one wants to use a different parameterization. For more details about curvature optimization in FOCUS see the paper titled "Constrained stellarator coil curvature optimization with FOCUS".

How to Use

To use the curvature objective functions, a weight, "weight_curv", needs to be set. This variable and all other variables in this section are set in the "*.input" file. No changes to the "*.focus" file are necessary. An integer named "penfun_curv" can be set to 1 or 2 and determines which penalty function is used. The value of κ_0 is set by the variable "k0". The values of α, β, γ, and σ are set by the variables "curv_alpha", "curv_beta", "curv_gamma", and "curv_sigma".