The toroidal magnetic flux of any cross-sections on the magnetic surface is constant. This can be used to constrain coil currents to avoid trival solutions.

[called by: solvers.]

General

To avoid trivial solutions, like when $I_i \to 0 \; \forall i$, $f_B \to 0$, it is sufficient to constrain the enclosed toroidal flux. If $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary, then the toroidal flux through any poloidal cross-sectional surfaces is constant. We include an objective function defined as

$$f_\Psi(\mathbf{X}) \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left(\frac{\Psi_\zeta \; - \; \Psi_o}{\Psi_o} \right)^2 d\zeta, \tag{1}$$

where the flux through a poloidal surface, $\mathcal{T}$, produced by cutting the boundary with plane $\zeta = const.$ is computed using Stokes' theorem,

$$\Psi_\zeta(\mathbf{X}) \equiv \int_{\mathcal{T}} \mathbf{B} \cdot d\mathbf{S} = \oint_{\partial \mathcal{T}} \mathbf{A} \cdot d\mathbf{l}. \tag{2}$$

Here $d\mathbf{l}$ is on the boundary curve of the poloidal surface and the total magnetic vector potential $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{X}) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N_C} I_i \int_{C_i} \frac{d\mathbf{l}_i}{r}. \tag{3}$$

The variation of f_Ψ resulting from $\delta \mathbf{x_i}$ is

$$\delta f_\Psi(\mathbf{X}) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\Psi_\zeta \; - \; \Psi_o}{\Psi_o} \right) \frac{\delta \Psi_\zeta}{\Psi_o} d\zeta, \tag{4}$$

where $\delta \Psi_\zeta = \int_{\partial \mathcal{T}} \delta \mathbf{A} \cdot d\mathbf{l}$ and

$$\delta \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[-\frac{\mathbf{r} \cdot \mathbf{x}_i'}{r^3} \; \delta \mathbf{x}_i \; + \; \frac{\mathbf{r} \cdot \delta \mathbf{x}_i}{r^3} \; \mathbf{x}_i' \right] dt. \tag{5}$$

First derivatives

We can write Eq.(5) into x, y, z components, (subscript i is omitting here)

$$\delta A_x = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta x dx + \Delta y dy + \Delta z dz}{r^3} \; x' - \frac{\Delta x x' + \Delta y y' + \Delta z z'}{r^3} \; dx \right] dt; \tag{6}$$

$$\delta A_y = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta x dx + \Delta y dy + \Delta z dz}{r^3} \; y' - \frac{\Delta x x' + \Delta y y' + \Delta z z'}{r^3} \; dy \right] dt; \tag{7}$$

$$\delta A_z = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta x dx + \Delta y dy + \Delta z dz}{r^3} \; z' - \frac{\Delta x x' + \Delta y y' + \Delta z z'}{r^3} \; dz \right] dt. \tag{8}$$

Here, we are applying $\mathbf{r} = \Delta x \; \mathbf{e_x}$ and $\mathbf{x}' = x' \mathbf{e_x}$. More specifically, $\Delta x = x_{surf} - x_{coil}$ and $x' = dx/dt$.
The first derivatives can be calculated as

$$\frac{\partial A_x}{\partial x} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[-\frac{\Delta y y' + \Delta z z'}{r^3} \right] dt; \tag{9}$$

$$\frac{\partial A_x}{\partial y} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta y x'}{r^3} \right] dt; \tag{10}$$

$$\frac{\partial A_x}{\partial z} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta z x'}{r^3} \right] dt. \tag{11}$$

$$\frac{\partial A_y}{\partial x} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta x y'}{r^3} \right] dt; \tag{12}$$

$$\frac{\partial A_y}{\partial y} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[-\frac{\Delta x x' + \Delta z z'}{r^3} \right] dt; \tag{13}$$

$$\frac{\partial A_y}{\partial z} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta z y'}{r^3} \right] dt. \tag{14}$$

$$\frac{\partial A_z}{\partial x} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta x z'}{r^3} \right] dt; \tag{15}$$

$$\frac{\partial A_z}{\partial y} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta y z'}{r^3} \right] dt; \tag{16}$$

$$\frac{\partial A_z}{\partial z} = \frac{\mu_0}{4\pi} I_i \int_0^{2\pi} \left[\frac{\Delta x x' + \Delta y y'}{r^3} \right] dt. \tag{17}$$