# B-Spline Basics - What you need to know for COIL_SPLINE implementation in STELLCOPT
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*The notes below are distilled from two good sources:*
*C-K Shene: https://pages.mtu.edu/~shene/COURSES/cs3621/NOTES/*
*and*
*https://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/node2.html*

- An order $k$ (degree $p$) B-spline curve is formed by joining several pieces of polynomials with at most $C^{k-2}$ continuity at the breaking points.

- A set of $m+1$ non-descending break points $t_0 \leq t_1 \leq \cdots \leq t_m$ defines a knot vector $\mathbf{T} = \{t_0, t_1, \cdots, t_m\}$. The $t_j$ are called knots and the range/scale of the $t_j$ is irrelevant. Conventionally the range is [0:1].

- A B-spline curve $\mathbf{C}(t)$ is a linear combination of control points $\mathbf{P}_i$ and B-spline basis functions $N_{i,k}(t)$ :

$$\mathbf{C}(t) = \sum_{i=0}^{n} N_{i,k}(t)\, \mathbf{P}_i \quad .$$

- It is required that $\quad m = n + k \quad \cdots$ The # of knots = the # of control points $+ k$

- Each knot span $\quad t_i \leq t \leq t_{i+1}\quad$ is mapped onto a polynomial curve between successive joints $\quad \mathbf{C}(t_i)\quad$ and $\quad \mathbf{C}(t_{i+1})$.

**End Point Control**

- Generally, B-spline curves do not pass through the two end control points

- However, if a knot has multiplicity $\mu = k - 1$ (it is repeated $\mu$ times) the curve will coincide with the control polygon at that knot.

- Repeating the knots at the end points $k - 1$ times will force the end points of the curve to coincide with the control polygon and to be tangent to the first and last legs of the control polyline. This is illustrated in the Fig below, stolen from **https://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/node2.html** where $n = 6$ and $m = 10$. The knot vector (assuming uniform spacing) would be

  **T** = 0, 0, 0, 0/4, 1/4, 2/4, 3/4, 4/4, 1, 1, 1
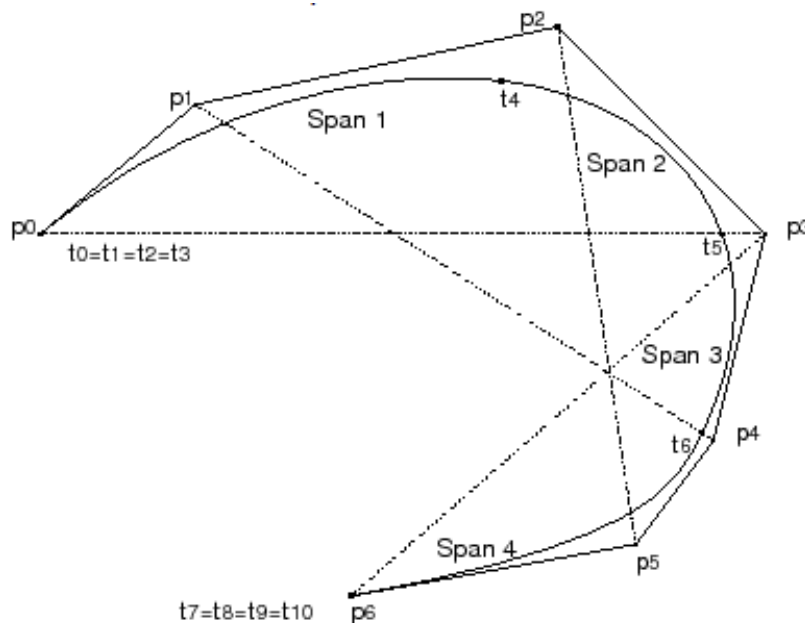
  This is called "coil clamping".



**Figure 1.11:** A clamped cubic B-spline curve

- This is important for our STELLCOPT application because there is a constraint on the beginning and end of each modular coil winding due to periodicity in real space. We must ensure that v at u=0 has the same value as v at u=1. This can be achieved for our cubic spline representation of the u-v coil curve by constraining $\mathbf{P}_0 = \mathbf{P}_n$ and choosing $\mu = 3$ at the knot end points $t_0$ and $t_m$ .

- Tangency to the control polygon at the end points follows from the fact that the first derivative of the B-spline curve is given by

$$\mathbf{C}'(t) = \sum_{i=0}^{n-1} N_{i+1,k-1}(t)\, \mathbf{Q}_i \qquad \text{where} \qquad \mathbf{Q}_i = \frac{k-1}{t_{i+k} - t_{i+1}}\, (\mathbf{P}_{i+1} - \mathbf{P}_i)$$

Therefore, the derivative of a B-spline curve is another B-spline curve of degree $k$ on the original knot vector with a new set of control points $\mathbf{Q}_0, \mathbf{Q}_1, \cdots, \mathbf{Q}_n$ .

Hence, $\mathbf{C}'(0) = \dfrac{k-1}{t_k - t_1}\, (\mathbf{P}_1 - \mathbf{P}_0)$ and $\mathbf{C}'(1) = \dfrac{k-1}{t_{n+k-1} - t_n}\, (\mathbf{P}_n - \mathbf{P}_{n-1})$ .

**Convex Hull and Local Support Property**

- A curve span lies within the convex hull of the control points that affect it. The i'th span of a cubic B-spline curve lies within the convex hull formed by control points $\mathbf{P}_{i-1}, \mathbf{P}_i, \mathbf{P}_{i+1}, \mathbf{P}_{i+2}$ .

- A single span of a B-spline curve is controlled only by $k$ control points, and any control point affects $k$ spans. Specifically, changing $\mathbf{P}_i$ affects the curve in the parameter range $t_i \leq t \leq t_{i+k}$ and a curve at a point $t$ where $t_r < t < t_{r+1}$ is completely determined by the control points $\mathbf{P}_{r-k+1}, \cdots, \mathbf{P}_r$ .
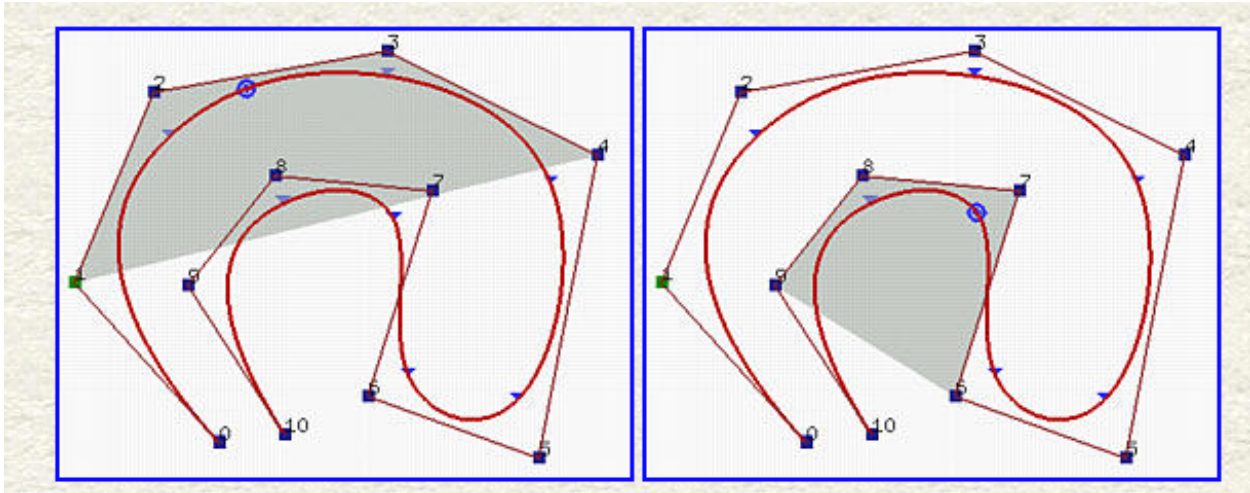
- In general, the lower the degree the closer a B-spline curve follows its control polyline.
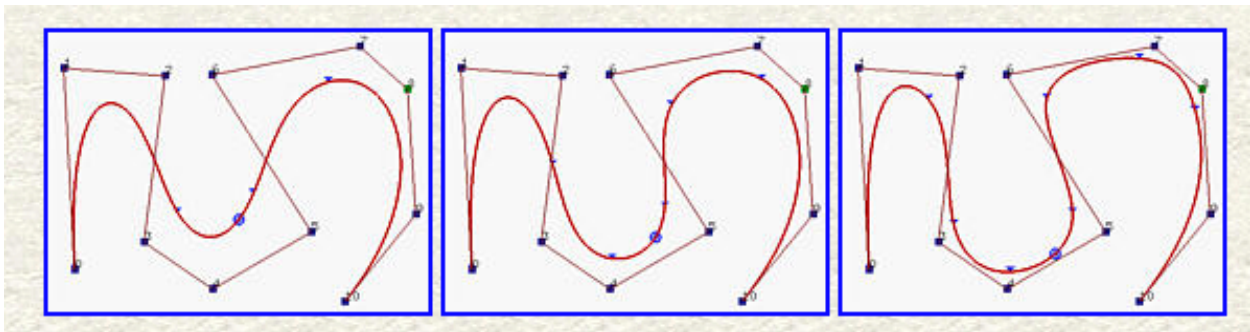


order k = 8
(degree p = 7)

order k = 6
(degree p = 5)

order k = 4
(degree p = 3)

**Force pure straightening of a curve segment by constraining $k$ adjacent control points to be collinear.**

- Shene's section titled "Moving Control Points" gives us the information we need to force pure straightening of a section of a B-spline curve.  On occasion we have used this in STELLCOPT to straighten the outleg section of MC coils to avoid interferences with blanket ports. The following (blue text and figures) are extracted from Shene's notes:

If $t$ lies in knot span $[t_i, t_{i+1})$, then $\mathbf{C}(t)$ lies in the convex hull defined by p+1 (i.e., k) control points $\mathbf{P}_i$, $\mathbf{P}_{i-1}$, $\cdots$ , $\mathbf{P}_{i-p+1}$, $\mathbf{P}_{i+p}$ . Since this is true for all $t$ in that span, the curve segment on this knot span lies entirely in this convex hull. If all of these $p$+1 control points are collinear (*i.e.*, on a straight line), the convex hull collapses to a line segment and so does the curve segment it contains. As a result, the curve segment on knot span $[t_i, t_{i+1})$ becomes a line segment. Note that in this case only this curve segment becomes a line segment. Other curve segments are still non-linear. Let us take a look at an example:



(a)   (b)   (c)

(d)   (e)   (f)
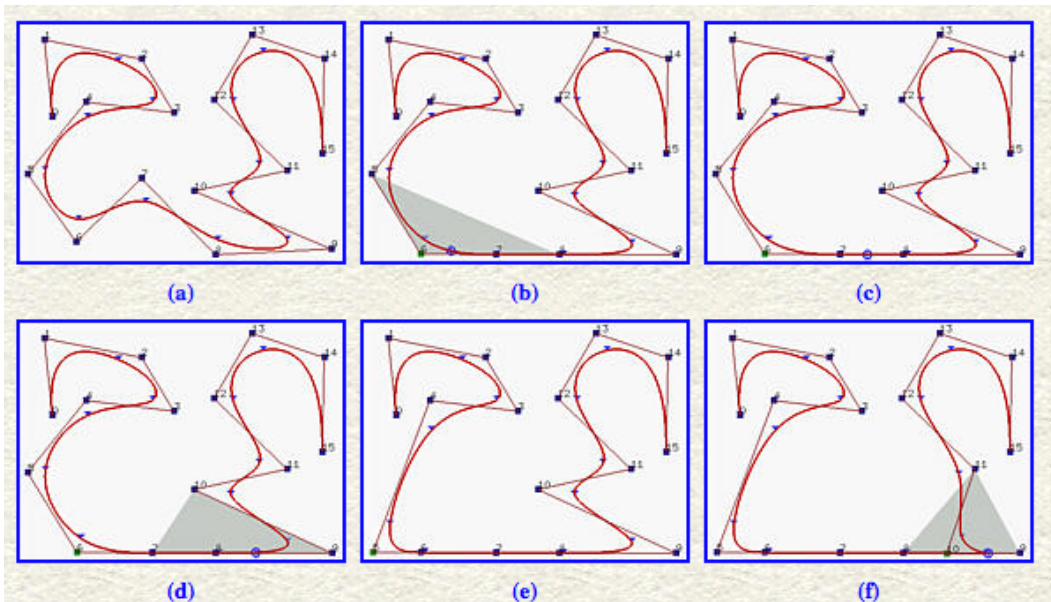
The figures above are obtained using $n = 15$ (*i.e.*, 16 control points), $k = 4$ (degree 3) and $m = 19$ (*i.e.*, 20 knots). The first four and last four knots are clamped. Figure (a) is the given B-spline curve. Let us make $P_9$, $P_8$, $P_7$ and $P_6$ collinear. Therefore, the curve segment on $[t_9, t_{10})$ lies in the convex hull defined by $P_9$, $P_8$, $P_7$ and $P_6$. Since this convex hull is a line segment, the curve segment must also be a line segment. Keep in mind that the first four knots are clamped and hence the first three knot spans do not exist. Since $[t_9, t_{10})$ is the seventh knot span, the seventh segment collapses to the line segment $P_7P_8$. This is illustrated by figures (b), (c) and (d).

But, why is the curve segment on $[t_9, t_{10})$ the only collapsed curve segment? Look at Figure (b). The shaded area is the convex hull just before $u$ enters $[t_9, t_{10})$. This convex hull is defined by control points $P_8$, $P_7$, $P_6$ and $P_5$, which is still not a line segment yet. Once $t$ enters $[t_9, t_{10})$, the curve segment collapses (Figure (c)). Immediately after $t$ leaves $[t_9, t_{10})$, a new convex hull appears (Figure (d)).

Figure (e) has $P_5$ collinear with its four successors. The curve contains one more line segment. Figure (f) has $P_{10}$ collinear with its five predecessors; however, it is moved to a position between $P_8$ and $P_9$. This would make part of the corresponding curve segment a straight line.

- To get a feel for the relationship between control points and curve shape, take a look at the B-spline demo at http://nurbscalculator.in The plot window on the right side shows a simple cubic B-spline curve (red) with 4 control points (the minimum number allowed for a cubic spline) in green and the polyline shown in black. First, click the XY Plane box showing below the plot window. This will ensure you are working with a curve in 2D (the relevant case in STELLCOPT)

  🔘 **Modify control point position**
  ⚪ **Add new control point**
  ⚪ **Remove control point**

  | ZoomToFit | XY Plane | YZ Plane | ZX Plane |

Click and drag one of the green control points. This is rather trivial, so now you can select "**Add new control point**" and click at a few positions (say 4) on the window (beginning somewhere near the top right control point - the curve starts at the bottom left, and ends at the top right as you can see from the Table of values on the left side of the screen. Before you do anything else select "**Modify control point position**". Now you have a less trivial shape you can modify by dragging one of  the control point location green boxes. In the lower of the Tables on the LHS, note the values of the knots (uniform) then click on the "**Clamped at start**" button and see what happens to the knot vector and to the shapoe of the red curve. Now click on the "**Clamped at end"** button and note the changes. Try moving some of the control points for this clamped curve. Finally, experiment with dragging a couple of the control points to enforce collinearity of three or more of the $\mathbf{P}_i$.  This should make you comfortable with the veracity of the notes in this write-up.

**Adding control points without changing the shape of a B-spline curve**

This is covered in Shene's notes under the topic of Knot Insertion. The topic is useful for STELLCOPT because we may run with a particular number of control points and find a coil design corresponding to an effective local minimum whose associated physics properties are not 'good enough'. So it is natural to want to check whether the calculation has not included enough degrees of freedom in the coil specification.

Given a coil curve described by n+1 control points, $\mathbf{C}(t) = \sum_{i=0}^{n} N_{i,k}(t)\ \mathbf{P}_i$ , we would

like to increase n but preserve the <u>identical</u> curve shape so that when we run STELLCOPT we are using exactly the same coil shapes and physics. We note that because of the relationship $m = n + k$, for every knot that is inserted for a given curve there will also be a control point inserted.

**Knot Insertion (for cubic B-splines):**

Consider a curve segment corresponding to the knot span $[t_k, t_{k+1})$ corresponding to the four control points $\mathbf{P}_{k-3}, \mathbf{P}_{k-2}, \mathbf{P}_{k-1}, \mathbf{P}_k$. These points are vertices of the polyline with three legs shown below.
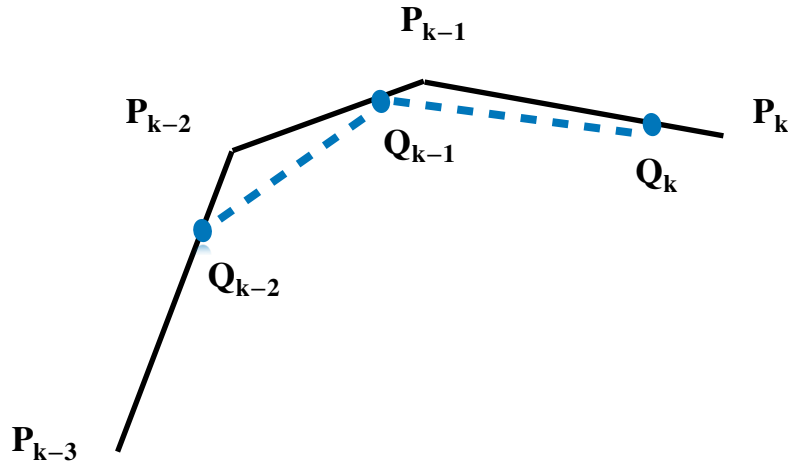


To insert a new knot $\tau$, we first find the knot span $[t_k, t_{k+1})$ that contains $\tau$.

For that $k$, three new control points $\mathbf{Q}_{k-2}, \mathbf{Q}_{k-1}, \mathbf{Q}_k$ are computed using

$$\mathbf{Q}_i = (1 - a_i)\mathbf{P}_{i-1} + a_i\mathbf{P}_i, \quad \text{with } \mathbf{Q}_i \text{ on leg } \mathbf{P}_{i-1}\mathbf{P}_i \text{, and where}$$

$$a_i = \frac{\tau - t_i}{t_{i+3} - t_i} \quad \text{for} \quad k - 2 \leq i \leq k.$$

Finally, the original polyline between $\mathbf{P}_{k-3}$ and $\mathbf{P}_k$ is replaced by the new polyline defined by $\mathbf{P}_{k-3}, \mathbf{Q}_{k-2}, \mathbf{Q}_{k-1}, \mathbf{Q}_k, \mathbf{P_k}$. After inserting the new knot, the knot vector becomes $t_0, t_1, \cdots, t_k, \tau, t_{k+1}, \cdots, t_m$.

FORTRAN code **/p/stellopt/ANALYSIS/Pomphrey/UTILITIES/add_knots.f** adds one or several control points to an existing B-spline curve. Below, we show an example of a coil curve taken from a STELLCOPT run using 15 control pts (n=14) and 19 equally spaced knots (m=18). We add 5 knots (and therefore 5 control points) in such a way that the coil shape is exactly preserved. We show only the Type C coil, but of course we can add control points to any other or all of the coils. The resulting spline curve would

form a starting point for a new STELLCOPT run with 6 additional degrees of fredom per coil.



15 and 21 control pts giving identical coil curve
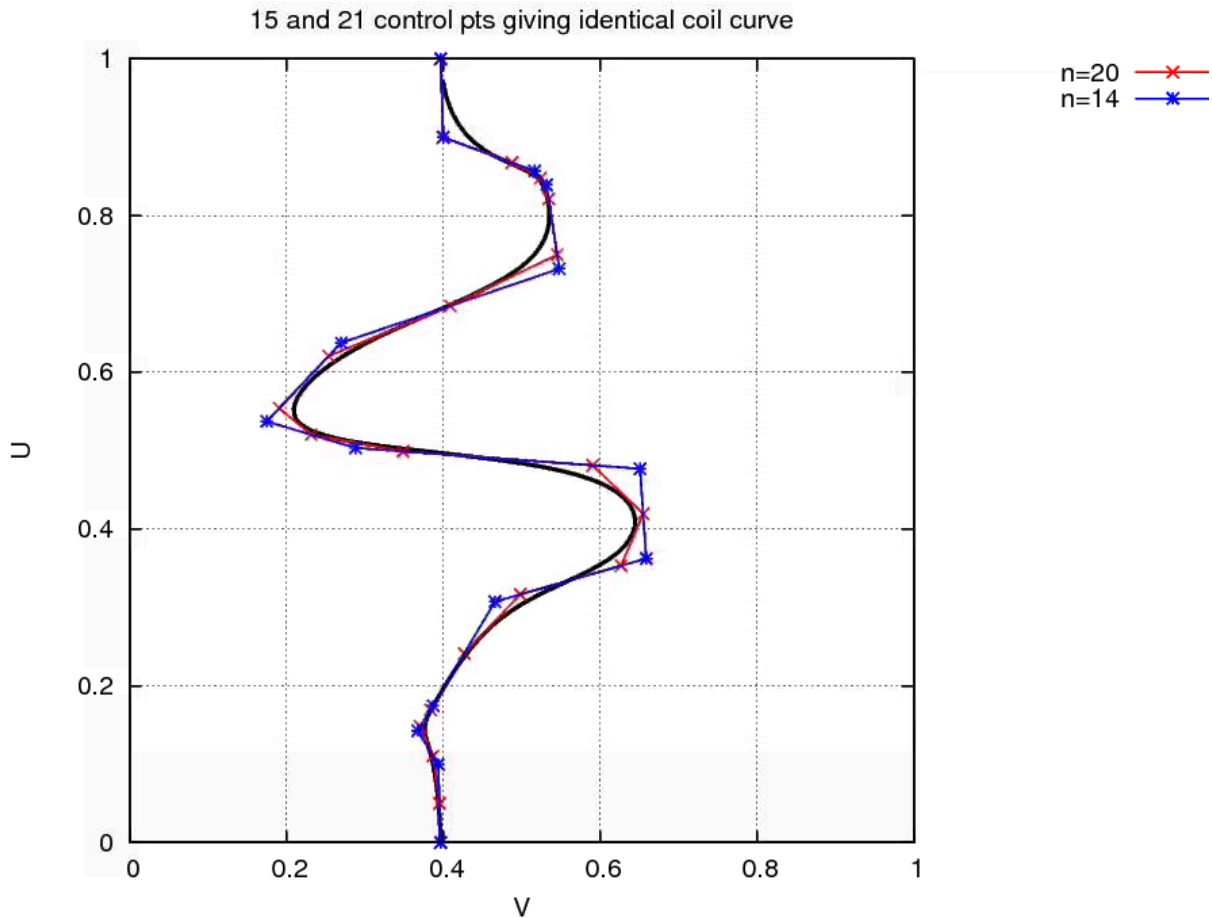
Polylines for the original n=14 and new n=20 B-splines are shown in blue and red, respectively. In this example we chose to sweep through the add_knots algorithm six times with $k$ = 3, 6, 9, 12, 15, and 18. This results in a change from the original 15 control point sequence of $\mathbf{P_0}, \mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}, \mathbf{P_4}, \mathbf{P_5}, \mathbf{P_6}, \mathbf{P_7}, \mathbf{P_8}, \mathbf{P_9}, \mathbf{P_{10}}, \mathbf{P_{11}}, \mathbf{P_{12}}, \mathbf{P_{13}}, \mathbf{P_{14}}$ to the new 21 point sequence $\mathbf{P_0}, \mathbf{Q_1}, \mathbf{Q_2}, \mathbf{Q_3}, \mathbf{P_3}, \mathbf{Q_4}, \mathbf{Q_5}, \mathbf{Q_6}, \mathbf{P_6}, \mathbf{Q_7}, \mathbf{Q_8}, \mathbf{Q_9}, \mathbf{P_9}, \mathbf{Q_{10}}, \mathbf{Q_{11}}, \mathbf{Q_{12}}, \mathbf{P_{12}}, \mathbf{Q_{13}}, \mathbf{Q_{14}}, \mathbf{Q_{15}}, \mathbf{P_{15}}$ preserving the boundary condition that the beginning and end points of the curve are the same point. The following tables may help keep track of curve segments:

Original Knots:

| $t_0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ | $t_9$ | $t_{10}$ | $t_{11}$ | $t_{12}$ | $t_{13}$ | $t_{14}$ | $t_{15}$ | $t_{16}$ | $t_{17}$ | $t_{18}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1/12 | 2/12 | 3/12 | 4/12 | 5/12 | 6/12 | 7/12 | 8/12 | 9/12 | 10/12 | 11/12 | 1 | 1 | 1 | 1 |

Knot spans: $[t_k, t_{k+1})$

| 3-4 | 4-5 | 5-6 | 6-7 | 7-8 | 8-9 | 9-10 | 10-11 | 11-12 | 12-13 | 13-14 | 14-15 |
|---|---|---|---|---|---|---|---|---|---|---|---|

Curve segments: $\mathbf{P}_k$ - $\mathbf{P}_{k+4}$ associated with above knot spans

| 0-3 | 1-4 | 2-5 | 3-6 | 4-7 | 5-8 | 6-9 | 7-10 | 8-11 | 9-12 | 10-13 | 11-14 |
|---|---|---|---|---|---|---|---|---|---|---|---|