BNORM CODE

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Given an arbitrary magnetic field $\mathbf{B}_0$ in a toroidal domain with $\mathbf{B}_0 \cdot \mathbf{n} = 0$ on the boundary $\mathbf{x} = \mathbf{x}(u, v)$ where $\mathbf{n}$ is the exterior normal and (u, v) are poloidal and toroidal angle-like variables.
Compute the normal component of the magnetic field produced by the surface current $\mathbf{j} = \mathbf{B}_0 \times \mathbf{n}$ on the boundary.

$$B_n = -\mathbf{n} \cdot \nabla \times \mathbf{A}, \tag{1}$$

where the vector potential is given by

$$\mathbf{A} = \frac{1}{4\pi} \int df' \; \frac{\mathbf{j}'}{\mid \mathbf{x} - \mathbf{x}' \mid}, \tag{2}$$

with

$$\mathbf{j} = \mathbf{B}_0 \times \mathbf{n}, \qquad\qquad \mathbf{n} = -\frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}, \tag{3}$$

$\mathbf{n}$ = the exterior normal, $\mathbf{x}_u := \frac{\partial \mathbf{x}}{\partial u}$.
Compute the vector potential on the boundary:
Inserting eq.(3) into eq.(2) one gets

$$\mathbf{A}(u, v) = \frac{1}{4\pi} \int du' dv' \; \frac{\mathbf{x}_{u'}(\mathbf{B}_0' \cdot \mathbf{x}_{v'}) - \mathbf{x}_{v'}(\mathbf{B}_0' \cdot \mathbf{x}_{u'})}{\mid \mathbf{x}(u, v) - \mathbf{x}(u', v') \mid} \tag{4}$$

The singularity is treated as follows: one introduces a function analytically integrable[1], periodic with respect to u' and v' and with the same singular behaviour as the above integrand.
Expanding the integrand at the singularity:

$$\frac{\mathbf{x}_{u'}(\mathbf{B}_0' \cdot \mathbf{x}_{v'}) - \mathbf{x}_{v'}(\mathbf{B}_0' \cdot \mathbf{x}_{u'})}{\mid \mathbf{x}(u, v) - \mathbf{x}(u', v') \mid} \approx \frac{\mathbf{x}_u(\mathbf{B}_0 \cdot \mathbf{x}_v) - \mathbf{x}_v(\mathbf{B}_0 \cdot \mathbf{x}_u)}{(g_{uu}\delta u^2 + 2\; g_{uv}\delta u \;\delta v + g_{vv}\;\delta v^2)^{1/2}} \tag{5}$$

with $\delta u = u' - u$, $\delta v = v' - v$ and $g_{uu} = \mathbf{x}_u \cdot \mathbf{x}_u$, $g_{uv} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{vv} = \mathbf{x}_v \cdot \mathbf{x}_v$.
An appropriate regularization integral is

$$I(a, b, c) = \pi \int_0^1 \int_0^1 \frac{du\; dv}{(a \tan^2(\pi u) + 2b \tan(\pi u) \tan(\pi v) + c \tan^2(\pi v))^{\frac{1}{2}}} \tag{6}$$

The integrand has the same singularity and the analytically performed integration gives

$$I(a,b,c) = T_0^+ + T_0^- \tag{7}$$

with

$$T_0^{\pm} = \frac{1}{\sqrt{a \pm 2b + c}} \log \frac{\sqrt{c(a \pm 2b + c)} + c \pm b}{\sqrt{a(a \pm 2b + c)} - a \mp b} \tag{8}$$

The expression for the vector potential can be written as

$$\mathbf{A}(u,v) = \mathbf{A}_{reg}(u,v) + \mathbf{A}_{sing}(u,v) \tag{9}$$

with

$$\begin{aligned} \mathbf{A}_{reg}(u,v) \quad = \quad & \frac{1}{4\pi} \int du' dv' \Big(\frac{\mathbf{x}_{u'}(\mathbf{B}_0' \cdot \mathbf{x}_{v'}) - \mathbf{x}_{v'}(\mathbf{B}_0' \cdot \mathbf{x}_{u'})}{\mid \mathbf{x}(u,v) - \mathbf{x}(u',v') \mid} \qquad (10) \\ & - \frac{\pi(\mathbf{x}_u(\mathbf{B}_0 \cdot \mathbf{x}_v) - \mathbf{x}_v(\mathbf{B}_0 \cdot \mathbf{x}_u))}{(g_{uu} \tan^2(\pi(u'-u)) + 2g_{uv} \tan(\pi(u'-u)) \tan(\pi(v'-v)) + g_{vv} \tan^2(\pi(v'-v))^{\frac{1}{2}}} \Big) \end{aligned}$$

and

$$\mathbf{A}_{sing}(u,v) \quad = \quad (\mathbf{x}_u(\mathbf{B}_0 \cdot \mathbf{x}_v) - \mathbf{x}_v(\mathbf{B}_0 \cdot \mathbf{x}_u)) \quad I(g_{uu}, g_{uv}, g_{vv}) \tag{11}$$

The integral for $\mathbf{A}_{reg}(u,v)$ can be performed numerically.
The eq.(1) for B_n can be written as

$$\begin{aligned} B_n &= -\frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|}(\mathbf{x}_u \times \mathbf{x}_v) \cdot (\nabla \times \mathbf{A}), \\ B_n &= \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|}(\mathbf{x}_v \cdot \frac{\partial \mathbf{A}}{\partial u} - \mathbf{x}_u \cdot \frac{\partial \mathbf{A}}{\partial v}), \\ B_n &= \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|}(\frac{\partial}{\partial u}(\mathbf{x}_v \cdot \mathbf{A}) - \frac{\partial}{\partial v}(\mathbf{x}_u \cdot \mathbf{A})). \qquad (12) \end{aligned}$$

The derivatives with respect to u, v are obtained by Fourier decomposing $\mathbf{x}_v \cdot \mathbf{A}, \mathbf{x}_u \cdot \mathbf{A}$

$$\begin{aligned} h(u,v) &= \int dudv \; \hat{h}(m,n) \quad cos(2\pi(mu+nv)), \qquad (13) \\ \frac{\partial h(u,v)}{\partial u} &= -\int dudv \; 2\pi m \; \hat{h}(m,n) \quad sin(2\pi(mu+nv)) \end{aligned}$$

[1] P. Merkel, J.Comput.Physics **66**,83(1986)